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On fluid motions induced by an electric current source

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In this paper we consider the flow field induced in an incompressible viscous conducting fluid, occupying the interior of a right circular cone, by an electric-current source situated at the vertex of the cone. We assume that the velocity field is small and its effect on the electromagnetic field is negligible. A similarity solution is obtained for the non-linear problem. This solution is an adaptation of Slezkin's solution for the momentum transfer through a viscous jet and, apart from the numerical solution of a Riccati type of equation, is exact. In particular, we investigate the case when the half angle of the cone is $\frac{1}{2}\pi$ and the fluid occupies the whole space on one side of an infinite plane. We also consider the corresponding inviscid flow problem that was recently investigated by another author and suggest that the solution obtained is not physically realistic.

1. Introduction

In some practical problems, such as in arc welding and electrochemistry, a current is passed through a conducting fluid. If the Lorentz force which is set up by the current and the associated magnetic field is rotational, a flow field will be induced. Recently Shercliff (1970), in an attempt to gain some understanding of the physics of the problem, investigated the flow field set up in an inviscid fluid by the Lorentz force due to a current source and the associated magnetic field. The current is supplied through a small hole (mathematically a point) of the wall bounding the semi-infinite region occupied by the fluid. Shercliff, by making the further assumption that the flow field is weak and its effect on the electromagnetic field is negligible, constructed a solution for this problem. The boundary conditions used by Shercliff make the velocity field have a singularity along the axis of the source. Shercliff suggests that it is necessary to choose these boundary conditions in order to make the vorticity finite near the wall. One might of course suggest a different and physically more realistic set of boundary conditions, namely finite velocity along the axis and large vorticity (due perhaps to a vortex sheet) near the wall. This suggested set of boundary conditions, however, makes the stream function of the flow field imaginary and thus does not give a solution.

It is not difficult to recognize that all the trouble is due to the assumption that the fluid is inviscid. This assumption reduces the order of the fundamental (momentum) equation of the flow field. Here we include viscosity in the momentum equation, though we still neglect the effect of the velocity on the electromagnetic field. The velocity field obtained is finite everywhere. It is, however, found that, when $J_0^2/\rho\nu^2$ (J_0 is the total current of the source, ρ the fluid density and ν the coefficient of kinematic viscosity) exceeds a certain magnitude, the velocity field has singularities.

2. Equations of the problem

We consider a uniform viscous incompressible conducting fluid, of density ρ and coefficient of kinematic viscosity ν, occupying the interior of a right circular cone. We use a spherical polar co-ordinate (r, θ, ϕ) system with the origin at the vertex of the cone and the line $\theta = 0$ along the axis of the cone. The generators of the cone are given by $\theta = \theta_0$. At the vertex of the cone there is a source supplying a current J_0 to the fluid. We assume that the induced flow field is small and its effect on the electromagnetic field is negligible; that is, we assume that the current is driven solely by the electrostatic field. We also assume that in the fluid the current is purely radial and symmetric with respect to the line $\theta = 0$; that is, we assume that the current density $\mathbf{j}$ is given by

$$\mathbf{j} = \hat{\mathbf{r}}\, J_0 f'(\mu)/2\pi r^2, \tag{1}$$

where $\mu = \cos\theta$ and a prime denotes differentiation with respect to μ. Since the total current is J_0, a constant, we must have

$$f(1) - f(\mu_0) = 1, \tag{2}$$

where $\mu_0 = \cos\theta_0$. On using the equation $\nabla\times\mathbf{B} = 4\pi\mathbf{j}$, we find that the associated magnetic field is given by

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \times 2J_0[f(1) - f(\mu)]/r(1-\mu^2)^{\frac{1}{2}}, \tag{3}$$

and is zero along the axis $\theta = 0$.

It is obvious, from the overall symmetry of the problem, that the velocity field is also symmetric about the axis $\theta = 0$. It is thus convenient to satisfy the equation of continuity by introducing a stream function ψ, so that the fluid velocity $\mathbf{v}$ is given by

$$\mathbf{v} = \left(-\frac{1}{r^2}\frac{\partial\psi}{\partial\mu},\quad -\frac{1}{r(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\psi}{\partial r},\quad 0\right). \tag{4}$$

On eliminating the pressure from the momentum equation our governing equation, in the steady state, becomes

$$\rho\nabla\times[(\nabla\times\mathbf{v})\times\mathbf{v}] = \nabla\times(\mathbf{j}\times\mathbf{B}) - \nu\rho\nabla\times\nabla\times(\nabla\times\mathbf{v}). \tag{5}$$

The only physical parameters appearing in this problem are J_0 which has dimensions $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$, ρ which has dimensions ML^{-3} and ν having dimensions L^2T^{-1}. Since the dimensions of ν are the same as those of $J_0/\rho^{\frac{1}{2}}$ we cannot form a fundamental time and a fundamental length parameter and we must find a solution by a similarity method. Thus we set

$$\psi = \nu r g(\mu, K), \tag{6}$$

where $K = 2J_0^2/\pi\rho\nu^2$.

The electrical conductivity σ of the fluid has the dimensions of $1/\nu$ and does not affect the above dimensional argument. σ, however, appears in the induction equation and shows when we can neglect the effect of the velocity on the electromagnetic field. From this equation we have

$$\nabla \times \nabla \times \mathbf{B} = 4\pi\sigma\nabla \times (\mathbf{v} \times \mathbf{B}).$$

The right-hand side of this equation is of order $\nu\sigma g$ relative to its left-hand side. Therefore the condition that the right-hand side of this equation, and the effect of the flow on the electromagnetic field are negligible, is

$$\nu\sigma g \ll 1.$$

Since the electric field and $\mathbf{j}$ are curl free we must also have $f'' = 0$. In view of (2)

$$f(\mu) = \mu/(1-\mu_0).$$

Equation (5) has only one component which is in the azimuthal direction. From (1), (3), (4) and (6) we find that this component is given by

$$\nu^2(1-\mu^2)^{\frac{1}{2}}(3g'g''+g'''g) = \nu^2(1-\mu^2)^{\frac{1}{2}}[(1-\mu^2)g^{\mathrm{iv}}-4\mu g'''] - \frac{\nu^2K[f(1)-f(\mu)]f'(\mu)}{(1-\mu^2)^{\frac{1}{2}}}$$

or
$$\frac{d^3}{d\mu^3}g^2 = 2(1-\mu^2)g^{\mathrm{iv}}-8\mu g'''-2K[f(1)-f(\mu)]f'(\mu)/(1-\mu^2). \tag{7}$$

When $K = 0$, (7) becomes identical with the equation occurring in the momentum transfer through a jet. The general solution of this equation, when $K = 0$, is

$$g^2 = 2(1-\mu^2)g'+4\mu g+a\mu^2+b\mu+c, \tag{8}$$

where a, b and c are arbitrary constants. This solution was obtained by Slezkin (1934) and has been discussed by several authors and reviewed by Whitham (1963, pp. 150–155) and by Batchelor (1967, pp. 205–211). The right-hand side of (7) is linear in g and thus the solution of (7) is

$$g^2 = 2(1-\mu^2)g'+4\mu g+a\mu^2+b\mu+c-2KF(\mu), \tag{9}$$

where $F(\mu)$ is the expression obtained by integrating $[f(1)-f(\mu)]f'(\mu)/(1-\mu^2)$ three times.

The velocity is zero when $\mu = \mu_0$ and therefore $g(\mu_0) = g'(\mu_0) = 0$. When $\mu = 1$ the velocity is finite and therefore $g(1) = 0$ and $g'(1)$ is *finite*. These boundary conditions require

$$a\mu_0^2+b\mu_0+c = 2KF(\mu_0); \quad a+b+c = 2KF(1); \quad 2a+b = 2KF'(1). \tag{10}$$

Equation (9) is a Riccati type of equation and by the substitution

$$g = -2(1-\mu^2)u'/u \tag{11}$$

is transformed into

$$u'' = \frac{u}{4(1-\mu^2)^2}[a\mu^2+b\mu+c-2KF(\mu)]. \tag{12}$$

The boundary conditions for the solution of (12) are

$$u(\mu_0) = \text{const.}, \quad \text{say } u(\mu_0) = 1, \quad \text{and} \quad u'(\mu_0) = 0. \tag{13}$$

When μ_0 is given, the solution of (12) is a straightforward job. The flow field is then computed from (6) and (4).

The pressure p, obtained by integrating the momentum equation, is given by

$$p = p_\infty + \nu^2 \rho P(\mu)/r^2,$$

where p_∞ is the pressure at infinity and

$$P(\mu) = -\frac{g^2}{2(1-\mu^2)} - g' - \tfrac{1}{2}a + \tfrac{1}{2}K F''(\mu). \tag{14}$$

This expression for the pressure, apart from the last term, which is due to the $\mathbf{j} \times \mathbf{B}$ force, is identical to the corresponding expression occurring in the momentum transfer through a jet [see equation (88) of Whitham's (1963) article].

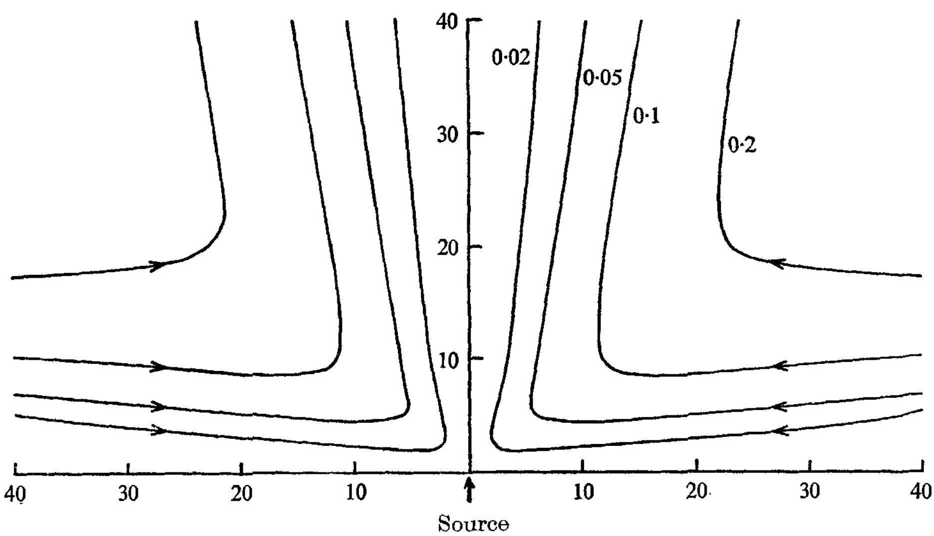

FIGURE 1. Streamlines for the case $K = 1$. The numbers on the streamlines are values of ψ measured in units of νL, where L is a characteristic length. The distances along the axes are in units of L.

3. Uniform current source through a plane infinite wall

In this case

$$\mu_0 = 0; \quad f(\mu) = \mu; \quad F(\mu) = \tfrac{1}{2}(1+\mu)^2 \log(1+\mu).$$

From the boundary conditions (10) we obtain

$$a = 2K, \quad b = (4\log 2 - 2)K, \quad c = 0$$

and therefore

$$u'' = \frac{Ku}{4(1-\mu^2)^2}[(4\log 2 - 2)\mu + 2\mu^2 - (1+\mu)^2 \log(1+\mu)]. \tag{15}$$

The coefficient of u in (15) is negative and is a slowly varying function of μ $(0 \leqslant \mu \leqslant 1)$. Its maximum absolute value is about $0{\cdot}0125K$. Thus, for small and moderate values of K, u'' is small and proportional to K. It follows then from (13) that so is u'. Thus, u decreases slowly as K increases and g/K is almost independent of K. This was confirmed by a numerical solution of (15) which showed that

$$u'/Ku, \tag{16}$$

and therefore g/K, increase very slowly as K increases. Percentagewise this increase is larger for larger μ. Thus, when μ is small the values of (16) for the cases when K is 10 and K is 0·01 are almost identical. Even around the point $\mu = 1$, the difference in the two sets of values of (16) corresponding to the cases $K = 0{\cdot}01$ and $K = 10$ is only about 2·7 %. From this discussion and (6) it follows that for small and moderate values of K the intensity of the flow field is proportional to

$$\nu K \sim J_0^2/\nu\rho, \quad \text{provided} \quad \nu\sigma g \ll 1.$$

Table 1 shows values of u, u', u'' and g for various values of μ for the case $K = 1$. Streamlines for the same value of K are shown in figure 1.

μ	u	$100u'$	$100u''$	$100g$
0	1·0000	0	0	0
0·1	1·0000	−0·0274	−0·4608	0·0489
0·2	0·9999	−0·0868	−0·7602	0·1667
0·3	0·9998	−0·1733	−0·9544	0·3154
0·4	0·9996	−0·2754	−1·0784	0·4628
0·5	0·9993	−0·3874	−1·1548	0·5815
0·6	0·9988	−0·5052	−1·1983	0·6475
0·7	0·9982	−0·6263	−1·2189	0·6399
0·8	0·9976	−0·7485	−1·2234	0·5402
0·9	0·9967	−0·8706	−1·2167	0·3319
1·0	0·9958	−0·9916	−1·2021	0

TABLE 1. Values of u, u', u'' and g for the case $K = 1$

As K increases, the coefficient of u on the right-hand side of (15) increases and u' and u'' increase in magnitude and u decreases. Thus when $K = 100$ the magnitude of (16) at $\mu = 1$ is about 39·2 % larger than that when $K = 0{\cdot}01$. If we continue to increase K a stage will be reached when $u(1) = 0$. A numerical solution of (15) shows that this occurs when K is about 300·1. This means that, when K is about 300·1, $g(1) \neq 0$ and our velocity field has a singularity on the axis. This, of course, is not realistic. Just before this breakdown occurs the velocity field is large and we cannot any more neglect its effect on the electromagnetic field. We think that, even when account is taken of the effect of the velocity on the electromagnetic field, there is a critical value of K, say K_c, which, if exceeded, makes the velocity field have singularities. It is possible that $K_c > 300{\cdot}1$.

Table 2 shows values of u, u', u'' and g for various values of μ for the case $K = 300$. The last columns of tables 1 and 2 show that $g(\mu)/K$ increases as K increases; that is, for a given fluid the intensity of the flow field increases faster than J_0^2. As explained above, this increase is larger near the axis of the source, and is more pronounced for large K. For small and moderate K, as K increases the increase in g/K is minimal, even near the axis of the source. Streamlines for the case $K = 300$ are shown in figure 2. Note the difference in the intensity and structure of the flow field near the axis of the current source between the cases for which $K = 1$ and $K = 300$.

μ	u	u'	u''	$100g/K$
0	1·0000	0	0	0
0·1	0·9974	−0·0740	−1·3790	0·0490
0·2	0·9815	−0·2586	−2·2386	0·1686
0·3	0·9436	−0·5086	−2·7020	0·3270
0·4	0·8788	−0·7883	−2·8444	0·5023
0·5	0·7859	−1·0687	−2·7246	0·6800
0·6	0·6658	−1·3263	−2·3965	0·8499
0·7	0·5220	−1·5428	−1·9121	1·0049
0·8	0·3591	−1·7052	−1·3212	1·1397
0·9	0·1830	−1·8051	−0·6703	1·2492
1·0	0·0003	−1·8386	−0·0011	0

TABLE 2. Values of u, u', u'' and g/K for the case $K = 300$

4. The inviscid fluid case

The problem of the last section was recently considered by Shercliff for the case when the fluid is inviscid. Below we briefly review Shercliff's solution and contrast it with our solution, in the limit as ν tends to zero.

When $\nu = 0$ we set
$$\psi_0 = K_0 r g_0(\mu),$$
where
$$K_0^2 = 2J_0^2/\pi\rho.$$
The equation satisfied by g_0 [equivalent to our equation (9)] is
$$g_0^2 = a_0\mu^2 + b_0\mu + c_0 - (1+\mu)^2 \log(1+\mu), \tag{17}$$

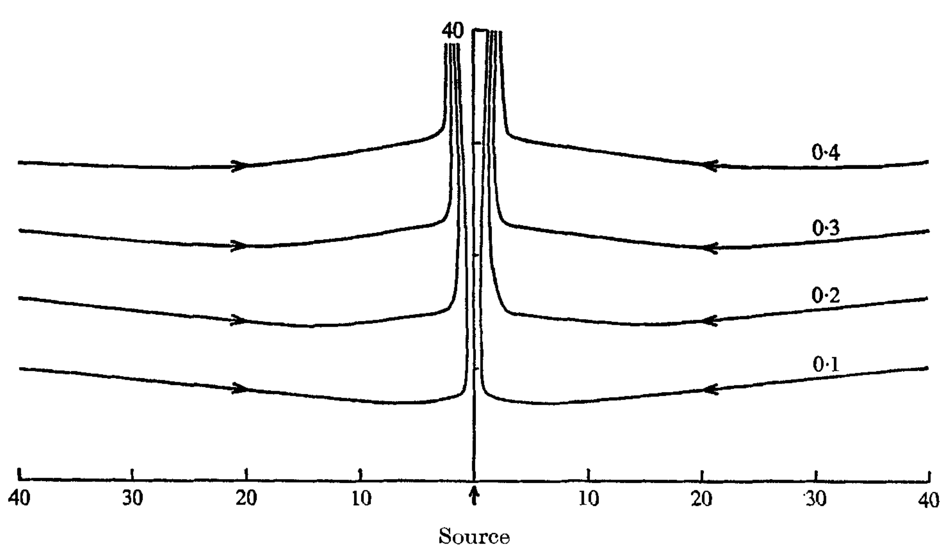

FIGURE 2. Streamlines for the case $K = 300$. The numbers on the streamlines are values of ψ/K measured in units of νL, where L is a characteristic length. The distances along the axes are in units of L.

where a_0, b_0 and c_0 are constants of integration. The pressure p is given by

$$p = p_\infty + \rho K_0^2 P(\mu)/r^2,$$

where p_∞ is the pressure at infinity. $P(\mu)$, obtained from the momentum equation, is given by

$$P(\mu) = -\frac{g_0^2}{2(1-\mu^2)} - \frac{1}{4}\frac{d^2}{d\mu^2}g_0^2.$$

On the plane $\mu = 0$, the normal component of the velocity is zero; that is, $g_0(0) = 0$ and therefore $c_0 = 0$. The velocity must be finite on the axis ($\mu = 1$); that is, $g_0(1)$ and g_0' are *finite*. This boundary condition requires that the right-hand side of (17) has a double zero at $\mu = 1$. Simple arithmetic shows that, if we satisfy this reasonable boundary condition as well, the right-hand side of (17) becomes negative and thus the problem has no solution. Shercliff, however, constructed a solution by modifying the boundary conditions. He made the normal component of the velocity zero on the plane $\mu = 0$ but he imposed the condition that the radial component of the velocity is finite near the plane $\mu = 0$; that is, he assumed that the right-hand side of (17) has a double zero at $\mu = 0$. He also assumed that the right-hand side of (17) had only one simple zero at $\mu = 1$. This modification implies, of course, that the radial component of the velocity is infinite along the axis of the source. There is no explanation, however, why we should have infinite velocity there. Indeed, the general question arises as to whether a solution that has singularities is permissible and, if so, whether Shercliff's inviscid flow solution is physically realistic. Note that in the momentum transfer through a fluid jet which was reviewed by Batchelor (1967) and by Whitham (1963) the velocity field is not allowed to have singularities except at the origin.

It would be instructive, in order to understand what happens to the solution when the viscosity is small, to look into our solution of the viscous flow problem as K increases and becomes larger than its critical value of 300·1. Obviously, the singularity moves away from the axis into the fluid [see equations (11) and (15)]. The singularity in the flow occurs, of course, along the generators of a cone having as axis the axis of symmetry of the problem. When K increases sufficiently and becomes, say, K_1, another singularity appears along the axis. When K becomes larger than K_1 this singularity also moves away from the axis into the fluid. When K is sufficiently larger than K_1 another singularity will appear on the axis which is eventually thrown into the fluid. Thus, as K increases indefinitely, our flow field has more and more singularities. The inviscid flow solution proposed by Shercliff suggests that, when K tends to infinity and ν tends to zero, $\nu g(\mu, K)\,[= K_0 g_0(\mu)]$ is finite everywhere except on the axis and all the singularities of the viscous flow problem are grouped on the axis. This is compensated by relaxing the viscous boundary conditions on the plane $\mu = 0$; that is, by not requiring $g'(0)$, in fact $\nu g'(0, K)\,[= K_0 g_0'(0)]$, to be zero.

In practice, however, we do not have inviscid fluids but fluids with small viscosity. If J_0^2/ρ is also small so that $2J_0^2/\pi\rho\nu^2 < 300{\cdot}1$ we must use the viscous flow solution. Note that the proposed inviscid flow solution suggests that the intensity of the flow field is proportional to K_0–that is, proportional to J_0–

whereas the viscous flow solution suggests that as J_0 increases the intensity of the flow field increases at least as fast as J_0^2. If ν is non-zero and $2J_0^2/\pi\rho\nu^2 > 300{\cdot}1$ the velocity field has singularities. If we ignore the viscosity effects altogether we are, in effect, as explained above, replacing all the singularities of the solution by one singularity along the axis of symmetry. We do not think that this approximation is justified and suggest that a solution based on this amalgamation of singularities is not a realistic one.

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